

Kinematics in Two Dimensions; Vectors

To express multidimensional motion requires a more complex manipulation of vectors than to express single-dimensional motion.

Vectors and Scalars

Quantities that express both magnitude and direction, such as displacement, velocity, and acceleration, are called **vectors**. Quantities that only express magnitude, such as time, speed, and distance traveled, are called **scalars**.

- Vectors are represented graphically by arrows whose points indicate direction. The lengths of vectors are proportional to their magnitudes within the same diagram.
- Vector variables are written in bold type, or with an arrow over italicized letters. Italicized letters without the arrow refer only to the magnitude of the vectors.

Addition of Vectors—Graphical Methods

If vectors are in the same direction, their summation requires only arithmetic in the manner of scalar addition. When vectors are not in the same direction, several methods are available for their addition.

- Graphically, two-dimensional motion can be represented by the path an object takes. As displacement measures only the net change in the position, the **resultant displacement** can be drawn from the starting position pointing to the finishing position.
- The **tail-to-tip method** for adding vectors requires successive vectors to be placed in the described manner, and the **resultant** drawn from the origin to the tip of the last vector. The resultant is independent of the order in which vectors are summed. As a vector sum, this is written as $\mathbf{D}_R = \Sigma \mathbf{D}$, although as the sum of their magnitudes, $D_R \leq \Sigma D$.
- The **parallelogram method** can be used to sum two vectors by placing both of their tails together and sketching the two remaining sides that would create a parallelogram. The diagonal with the common tail represents the resultant, whose direction is away from the initial vector's common origin.

Subtraction of Vectors, and Multiplication of a Vector by a Scalar

- Multiplication of a vector, \mathbf{V} , by a scalar, c , produces a vector with a magnitude of cV in the direction of \mathbf{V} . If c is a negative value, the vector $c\mathbf{V}$ has a magnitude of cV in opposite direction of \mathbf{V} .

-
- Subtracting two vectors can be viewed as summing two vectors where the formerly subtracted vector is considered as having been multiplied by the scalar -1 , thereby reversing its position. The tail-to-tip method of summation can then be applied.

Adding Vectors by Components

The projection of a vector onto perpendicular axes is called **resolving the vector into components**.

- The components of each vector are trigonometric functions of the angle that the vector makes with the positive x axis when the tail of a vector is based at the origin, such that $V_x = V \cos \theta$ and $V_y = V \sin \theta$. Further, $V = \sqrt{(V_x)^2 + (V_y)^2}$ and $\tan \theta = (V_y/V_x)$.
- Vectors can be added by summing the **components** of their respective axes.
- Coordinate axes can be placed to minimize computation.

Projectile Motion

Near the Earth's surface, an object that has a horizontal velocity vector component and is subject to gravity experiences **projectile motion**.

- The vertical component of an object projected horizontally is the same as the vertical component of an object experiencing free-falling motion.
- More generally, regardless of the angle of projection, the velocity vector can be resolved into its components.
- After an object is projected, the horizontal velocity component remains constant, while the vertical velocity component changes due to gravity.

Solving Problems Involving Projectile Motion

The basic kinematics equations can be applied using the components of projectile motion.

- The horizontal velocity is assumed to remain constant, and $x = x_0 + v_{x0}t$.
- For the vertical component, if upward displacement is defined as positive, $v_y = v_{y0} - gt$, $y = y_0 + v_{y0}t - 1/2(gt^2)$, and $v_y^2 = v_{y0}^2 - 2g(y - y_0)$.

Projectile Motion Is Parabolic

For all projectile motion, the horizontal and vertical components can be rearranged into a single equation with y as a function of x of the form $y = ax^2 + bx + c$, which is a quadratic equation graphically represented by a parabola. Since the value of a is negative, the parabola is concave downward as expected.

Relative Velocity

Calculations for motion observed from a moving reference require additional steps beyond those needed for motion observed from rest. Vector addition is necessary to compute the **relative velocity** of motion relative to such reference frames.

For Additional Review

If two-dimensional vectors can be expressed by their magnitude and a reference angle, think about an equivalent method of expressing three-dimensional vectors and how that would affect vector summation.

Multiple-Choice Questions

1. What is the vertical component of a 33 m vector that is at a 76° angle with the x axis?
(A) 25 m
(B) 28 m
(C) 30 m
(D) 32 m
(E) 33 m
2. A possible angle that a vector with a magnitude of 17 m could make with the x axis so that its horizontal component is 12 m is
(A) 15°
(B) 30°
(C) 45°
(D) 60°
(E) 75°
3. A ball is thrown horizontally out the window of a building with a velocity of 8.0 m/s from a height of 2500 m. How far from the base of the building will the ball land?
(A) 54 m
(B) 60 m
(C) 90 m
(D) 115 m
(E) 180 m
4. What vector represents the displacement of a person who walks 15 km at 45° south of east, then 30 km due west?
(A) 17° west of south
(B) 29° south of west
(C) 45° west of north
(D) 61° east of south
(E) 71° south of west
5. At its maximum height, the vertical component of velocity for an object projected at 4.9 m/s at a 60° angle is

- (A) 9.8 m/s
- (B) 4.9 m/s
- (C) 0 m/s
- (D) -4.9 m/s
- (E) -9.8 m/s

Questions 6–8

A plane is heading to a destination 1750 km due north at 175 km/h in a westward wind blowing 25 km/h.

6. At what angle from north should the plane be oriented so that it reaches its destination?
(A) 8°
(B) 15°
(C) 21°
(D) 35°
(E) 41°
7. Without the compensatory angle described in question 6, how far off course will the plane be after it has traveled 1750 km north?
(A) 150 km
(B) 175 km
(C) 250 km
(D) 375 km
(E) 675 km
8. Without the compensatory angle described in question 6, what will be the magnitude of the velocity vector?
(A) 151 km/h
(B) 164 km/h
(C) 170 km/h
(D) 175 km/h
(E) 177 km/h

9. At what velocity should a ball be thrown at a 45° angle so that it hits a target that is 20 meters away and placed at the same height?
- (A) 3 m/s
 (B) 7 m/s
 (C) 11 m/s
 (D) 14 m/s
 (E) 18 m/s
10. A bullet fired horizontally from a height of three meters with a velocity of 120 m/s will hit the ground after how many seconds?
- (A) 0.55 s
 (B) 0.78 s
 (C) 0.96 s
 (D) 1.6 s
 (E) 2.8 s

Free-Response Questions

- A projectile is fired at 27.5 m/s at a 60° angle from the ground.
 - What will be its maximum height?
 - How far will it travel horizontally?
 - At what magnitude of velocity should the projectile be fired from this angle so that it hits a target 100 meters away?
- A torpedo leaves a ship at 2.1 m/s initially aimed at a target 85 meters away. The current of the ocean flows perpendicular to this motion at 0.9 m/s.
 - How far off course will the torpedo be when it reaches 85 meters in the direction of the torpedo's initial velocity?
 - What will be the magnitude of the velocity vector that takes both the torpedo's velocity and the current into account?
 - At what angle will the resultant velocity be given with the torpedo's initial velocity?
 - At what angle should the torpedo be aimed to compensate for the current?

ANSWERS AND EXPLANATIONS

Multiple-Choice Questions

- (D) is correct.** The vertical component of the vector is given by $V_y = V \sin \theta = (33 \text{ m})(\sin 76^\circ) = 32 \text{ m}$.
- (C) is correct.** The horizontal component of the vector is given by $V_x = V \cos \theta$, where $V_x = 12 \text{ m}$ and $V = 17 \text{ m}$. The relation can be rearranged so that $\theta = \cos^{-1}(V_x/V) = \cos^{-1}(12/17) = 45^\circ$.
- (E) is correct.** The vertical position is given by $y = y_0 + v_{y0}t - 1/2(gt^2)$, where $y = 0$, $y_0 = 2500 \text{ m}$, $v_{y0} = 0$. So, $0 = 2500 \text{ m} - 4.9t^2$, so $t = 22.6 \text{ s}$. The horizontal position is given by $x = x_0 + v_{x0}t = 0 + (8 \text{ m/s})(22.6 \text{ s}) = 180 \text{ m}$.
- (B) is correct.** Both components of the first leg of the trip are $15 \sin 45^\circ = 15 \cos 45^\circ = 10.6 \text{ m}$, so 10.6 m south and 10.6 m east. The next leg of the trip is 30 miles west, so the net horizontal displacement is 19.4 m west, and the net vertical displacement is 10.6 m south. By the equation $V = \sqrt{(V_x^2 + V_y^2)}$, $V = 22.1 \text{ m}$ and $\theta = \tan^{-1}(V_y/V_x) = 29^\circ$ south of west.
- (C) is the correct answer.** At its maximum height, the velocity vector is equal to the horizontal component, and the vertical component is 0 m/s at that instant.

- **6. (A) is correct.** Because the velocities given are perpendicular, the plane should be angled such that $\sin \theta = (\text{wind velocity vector})/(\text{plane velocity vector}) = (25 \text{ km/h})/(175 \text{ km/h}) = 1/7$, so $\theta = 8^\circ$.
- **7. (C) is correct.** At 175 km/h, it will take 10 hours to travel 1750 km. In 10 hours, the wind blows 25 km/h to the west, so the plane will be off course by 250 km.
- **8. (E) is correct.** The magnitude of the resultant velocity will be given by $\sqrt{(V_x^2 + V_y^2)} = \sqrt{((175 \text{ km/h})^2 + (25 \text{ km/h})^2)} = 177 \text{ km/h}$.
- **9. (D) is correct.** The velocity sought has an x component of $V_x = V \cos \theta = V \cos 45^\circ = 0.707 V$ and a y component of $V_y = V \sin \theta = V \sin 45^\circ = 0.707 V$. Using the equations $x = x_0 + v_{x0}t$ and $y = y_0 + v_{y0}t - 1/2(gt^2)$, $20 = 0 + (0.707 V) t$, so $t = 20/0.707 V$, which can be plugged into $0 = 0 + (0.707 V) t - 1/2(gt^2) = (0.707 V)(20/0.707 V) - 4.9(20/0.707 V)^2$. Solving for V , $V = 14 \text{ m/s}$.
- **10. (B) is the correct answer.** The vertical component is given by $y = y_0 + v_{y0}t - 1/2(gt^2)$, where $y = 0$, $y_0 = 3 \text{ m}$, $v_{y0} = 0$. So, $0 = 3 \text{ m} - 4.9t^2$, so $t = 0.78 \text{ s}$.

Free-Response Questions

1. For parts (a) and (b), the components of velocity are $V_x = V \cos \theta = (27.5 \text{ m/s}) \cos 60^\circ = 13.75 \text{ m/s}$ and $V_y = V \sin \theta = 23.8 \text{ m/s}$.
- (a) The maximum height will occur when the vertical component of velocity is 0. From the equation, $v_y^2 = v_{y0}^2 - 2gy$; $(0 \text{ m/s})^2 = (23.8 \text{ m/s})^2 - (19.6 \text{ m/s}^2)y$, so $y = 28.9 \text{ m}$.
- (b) The projectile will be in the air for t seconds, where t satisfies $y = y_0 + v_{y0}t - 1/2(gt^2)$, given that the starting and finishing height is 0 m. $0 = (23.8 \text{ m/s}) t - (4.9 \text{ m/s}^2)t^2 = t[(23.8 \text{ m/s}) - (4.9 \text{ m/s}^2)t]$, so $t = 4.9 \text{ s}$. In that time, the projectile travels x meters, where $x = x_0 + v_{x0}t = 0 + (13.75 \text{ m/s})(4.9 \text{ s}) = 67 \text{ m}$.
- (c) The velocity sought has an x component of $V_x = V \cos \theta = V \cos 60^\circ = 0.5 V$ and a y component of $V_y = V \sin \theta = V \sin 60^\circ = 0.87 V$. Using the equations $x = x_0 + v_{x0}t$ and $y = y_0 + v_{y0}t - 1/2(gt^2)$, $100 \text{ m} = 0 \text{ m} + (0.5 V) t$, so $t = 100 \text{ m}/0.5 V$, which can be plugged into $0 \text{ m} = 0 \text{ m} + (0.87 V)t - 1/2(gt^2) = (0.87 V)(100 \text{ m}/0.5 V) - 4.9(100 \text{ m}/0.5 V)^2$. Solving for V , $V = 33.6 \text{ m/s}$.

This response utilizes most aspects of projectile motion. For (a), (b), and (c), the response correctly demonstrates breaking a vector into components for both numerical and variable velocity vectors. The vertical component is correctly used in (a) to determine maximum height, while the horizontal component is used in (b) to determine the horizontal displacement. By necessity, two kinematics equations are utilized consecutively in (b). In (c), these same techniques are demonstrated using a variable term throughout the calculations.

-
2. (a) At 2.1 m/s, it will take the torpedo 40.5 seconds to travel 85 m. In 40.5 seconds, the perpendicular current 0.9 m/s will carry the torpedo 36.4 m downstream from the target.
- (b) The magnitude of the resultant velocity will be given by $\sqrt{V_x^2 + V_y^2} = \sqrt{(2.1 \text{ m/s})^2 + (0.9 \text{ m/s})^2} = 2.3 \text{ m/s}$.
- (c) Because the velocities given are perpendicular, the torpedo's angle will be given by $\tan \theta = (\text{current velocity vector})/(\text{torpedo velocity vector}) = (0.9 \text{ m/s})/(2.1 \text{ m/s}) = 0.43$, so $\theta = 23^\circ$.
- (d) Because the velocities given are perpendicular, the torpedo should be aimed at the angle given by $\sin \theta = (\text{current velocity vector})/(\text{torpedo velocity vector}) = (0.9 \text{ m/s})/(2.1 \text{ m/s}) = 0.43$, so $\theta = 25^\circ$.

This response correctly accounts for the relative velocity of two perpendicular velocity vectors. The response to part a demonstrates a basic kinematics equation, while the response to part b utilizes the Pythagorean theorem to determine the magnitude of the resultant vector, which is the hypotenuse of the vector sum. The responses to parts c and d utilize the correct trigonometric relationships for the given velocity vectors.