CHAPTER 5

Circular Motion; Gravitation

Kinematics of Uniform Circular Motion

Uniform circular motion describes the movement of an object in a circular path at a constant speed, v, but at a constantly changing velocity vector, $\mathbf{v}$, necessary for the object to stay on the path. The velocity vector at a given point in the path is the tangent to the circular path in the direction of motion.

- The acceleration corresponding to this changing velocity at a given point is directed toward the center of the circle, and it is called **centripetal** or **radial acceleration.** Radial acceleration is proportional to the square of the velocity and inversely proportional to the radius, $a_R = v^2/r$.
- At a point in the path, the velocity and acceleration of an object undergoing uniform circular motion are perpendicular.
- The number of revolutions per second by the object is denoted by the **frequency,** f. The time in seconds for each revolution, T, is its reciprocal, $T = 1/f$, and is called the **period.**
- From these definitions, an equation for speed can be derived, $v = 2\pi r/T$.

Dynamics of Uniform Circular Motion

The force required to keep an object in uniform circular motion is directed inward toward the center and is the product of mass and radial acceleration, $\Sigma F_R = mv^2/r$.

- Were the force to stop being applied, the object would continue in the path of the velocity vector at that point, tangent to the circle of motion.
- Often the force is provided by the tension of a rope or string.
- Automobiles require friction between the wheels and the road to provide the inward force necessary for moving in a circle, or more often, in an arc of a circle.
- Satellites use gravity to provide the centripetal force for circular orbits.

Nonuniform Circular Motion

When the net force on an object is not directed toward the center, the force vector can be broken into two perpendicular components: the inwardly directed radial force, F_R, and the tangential force, F_{tan}. Similarly, the acceleration can be broken into two perpendicular components: its inward radial acceleration, a_R, and the tangential acceleration, a_{tan}.

- The change of speed in a nonuniform circular motion is a consequence of the tangential force, and thus of the tangential acceleration.
- When the speed of circular motion decreases, the velocity vector is antiparallel to the tangential acceleration component. When the speed of circular motion increases, the velocity vector is parallel to the tangential acceleration component.
- From the Pythagorean theorem, the magnitude of acceleration at a point is equal to the square root of the sum of the squares of the components, $a = \sqrt{a_R^2 + a_{tan}^2}$.

Newton's Law of Universal Gravitation

Newton derived the theory of gravitation with the goal of explaining the motion of the moon around the Earth. He reasoned that the radial force that keeps the Moon moving in an approximately uniform circular motion is related to the force of gravity observed at the Earth's surface.

- Newton's **law of universal gravitation** states that for both masses, the magnitude of the attractive force between them is proportional to the product of the masses and inversely proportional to the square of distance. This is now represented by $F = Gm_1m_2/r^2$, where G is the universal constant, 6.67×10^{-11} $N \cdot m^2/kg^2$, whose value was first approximated by Cavendish a century after Newton proposed his law.

Gravity Near the Earth's Surface; Geophysical Applications

Newton's law of universal gravitation is consistent with the equation $F = mg$ at the Earth's surface, as g represents the product of mass of the Earth, 5.98×10^{24} kg, and the constant G divided by the square of average radius of the Earth, 6.38×10^6 m. Values of g fluctuate slightly across the Earth's surface because there is not a single fixed radius and because the distribution of mass in the Earth is not uniform.

Satellites and "Weightlessness"

Satellites remain in orbit because their tangential speed is sufficient to keep them from falling to the Earth's surface, but not so great as to cause them to fly out of their circular or elliptic path. The inward-directed force and acceleration necessary for circular motion is due to gravity.

- Apparent changes in the force of gravity are influenced by the frame of reference of the observer. That is, if the frame of reference is accelerating in the same direction as gravity, apparent weight will decrease. However, if the frame of reference is accelerating in the opposite direction, the apparent weight will increase.
- While a satellite orbits the Earth, apparent weight on the satellite is decreased due to the satellite's acceleration. The phenomenon of weightlessness occurs because the frame of reference (the satellite) has a centripetal acceleration caused by gravity. A passenger on the satellite feels weightless, just as he or she would inside a freely falling elevator.

Kepler's Laws and Newton's Synthesis

Newton's work confirmed Kepler's laws, which had been developed earlier.

- **Kepler's first law** states that planetary pathways are elliptical, orbiting around the Sun, which is at one focus of the ellipse.
- **Kepler's second law** states that an imaginary line from the sun to the planet sweeps out equal areas in equal periods of time.
- **Kepler's third law** states that the ratio of the cube of a planet's mean distance from the Sun to the square of its period is constant for all planets. That is, for any two planets, $R_1^3/T_1^2 = R_2^3/T_2^2$ = constant.
- Kepler's third law is applicable to any two objects orbiting the same third object.
- Orbits of planets are not exactly elliptical, due to the forces of the other planets and their satellites, and the deviations from an elliptical orbit are called **perturbations.**

For Additional Review

Analyze the forces operating on a mass uniformly revolving in a circle whose plane is parallel to the force of gravity.

Multiple-Choice Questions

1. What is the speed of a moon with a period of 20 hours that is orbiting a planet of radius R at a distance 5 R from the planet's center?
   (A) $26{,}000/\pi R$ m/s
   (B) $\pi R/36{,}000$ m/s
   (C) $R/3{,}600\pi$ m/s
   (D) $\pi R/7{,}200$ m/s
   (E) $\pi^2 R/720$ m/s

2. What is the period of an object in uniform circular motion with a radius of 25 cm with a centripetal acceleration of 35 m/s^2?
   (A) 0.53 s
   (B) 3.1 s
   (C) 6.6 s
   (D) 12 s
   (E) 33 s

3. In which of the following situations can the velocity and acceleration vectors be perpendicular?
   I. An object falling vertically
   II. An object in uniform circular motion
   III. An object in projectile motion
   (A) I only
   (B) II only
   (C) III only
   (D) I and II only
   (E) II and III only

4. What is the centripetal acceleration of an object in uniform circular motion at the end of a 1 meter rope making 1 revolution per second?
   (A) 39 m/s^2 tangentially
   (B) 39 m/s^2 inward
   (C) 39 m/s^2 outward
   (D) 1 m/s^2 inward
   (E) 1 m/s^2 tangentially

5. In an ideal situation, the force required to keep a 20 gram sphere turning at 15 revolutions per second at the end of a 35 cm string of negligible mass is
   (A) 17 N
   (B) 45 N
   (C) 62 N
   (D) 121 N
   (E) 515 N

6. What is the attractive force between the Earth and an 85 kg piece of debris in an orbit of radius 27,500 km with two revolutions per day?
   (A) 1.1 N
   (B) 5.2 N
   (C) 17 N
   (D) 45 N
   (E) 121 N

7. At what distance will two 1000 kg masses have the same attractive force for each other as gravity at the surface of the Earth would have on either one of them?
   (A) 1.1×10^{0} m
   (B) 9.8×10^{-1} m
   (C) 2.9×10^{-3} m
   (D) 8.3×10^{-5} m
   (E) 4.5×10^{-7} m

8. What would be the value of the acceleration of gravity near the surface of the Earth if the Earth had the same mass and a radius of 6.6×10^{6} m?
   (A) 9.16 m/s^2
   (B) 9.31 m/s^2
   (C) 9.77 m/s^2
   (D) 9.9 m/s^2
   (E) 10.3 m/s^2

9. What is the period of a 1500 kg satellite in a circular orbit 48,000 km from the center of the Earth?
   (A) 24 hours
   (B) 26 hours
   (C) 29 hours
   (D) 34 hours
   (E) 51 hours

10. What is the frequency of a planet around a star in orbit at a mean distance of *R*, if it is twice the mean distance of a planet orbiting the same star with a frequency of 2.10×10^{-5} rev/s?
   (A) 1.4×10^{-5} rev/s
   (B) 7.4×10^{-6} rev/s
   (C) 1.7×10^{-7} rev/s
   (D) 6.8×10^{-7} rev/s
   (E) 2.4×10^{-8} rev/s

Free-Response Questions

1. Where friction allows, the movement of a turning car can resemble uniform circular motion.
   (a) Describe what happens to a 750 kg car as it travels an icy road that approximates the arc of a circle of radius 35 m at velocity, *V*, and can be assumed to briefly have uniform circular motion.
   (b) What is the coefficient of friction necessary to keep the car from sliding?
   (c) If the coefficient of friction is 0.45, what speeds would be safe for the car to travel?

2. Given the orbital radius and period of a satellite orbiting the Earth, what minimum information about the Moon would be necessary for Kepler's laws and Newton's laws to be used to confirm the mass of the Earth based on its relationship to the Moon?

ANSWERS AND EXPLANATIONS

Multiple-Choice Questions

1. (D) is correct. A period of 20 hours = (20 h)(60 min/h)(60 s/min) = 72,000 s. For uniform circular motion, velocity is given by $v = 2\pi r/T = 2\pi 5R/72{,}000$ m/s $= 10\pi R/72{,}000$ m/s $= \pi R/7{,}200$ m/s.

2. (A) is correct. The relationship between centripetal acceleration and period of circular motion involves the equations $a_R = v^2/r$ and $v = 2\pi r/T$, such that $a_R = (2\pi r/T)^2/r$ and $T = \sqrt{[(2\pi r)^2/r(a_R)]} = \sqrt{[(4\pi^2 r/(a_R)]} = \sqrt{[(4\pi^2(0.25\text{ m})/(35\text{ m/s}^2)]} = \sqrt{[(\pi^2 m/(35\text{ m/s}^2)]} = 0.53$ s.

3. (E) is the correct answer. In uniform circular motion, the acceleration and velocity vectors are perpendicular. In projectile motion, when an object reaches its maximum altitude (the maximum of the parabolic path), its acceleration and velocity vectors are perpendicular. When an object is falling vertically, its acceleration and velocity vectors are parallel.

4. (B) is correct. First, centripetal acceleration for uniform circular motion is inward, which narrows down the answer choices. Centripetal acceleration, $a_R = v^2/r$ where $v = 2\pi r/T$, $a_R = (2\pi r/T)^2/r = (2\pi)^2 = 39\text{ m/s}^2$.

5. (C) is correct. $f = 15$ rev/s is equivalent to $T = .067$ seconds per revolution. Given the force equation for uniform circular motion, $\Sigma F_R = mv^2/r =$ $(0.02\text{ kg})(2\pi r/T)^2/(0.35\text{ m}) = (0.02\text{ kg})(2\pi(0.35\text{ m})/0.067)^2/(0.35\text{ m}) = 62\text{ N}$.

6. (D) is correct. The attractive force between the Earth and the debris is given by $F = Gm_e m_d/r^2$
$= (6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\text{ kg})(85\text{ kg})/(2.75 \times 10^7\text{ m})^2 = 45\text{ N}$.
Question 6 could also be solved using the equations for uniform circular motion.

7. (D) is correct. The value for r for which $Gm_1m_2/r^2 = m_1 g$ is sought. Since $m_1 = m_2$, $r = \sqrt{Gm/g} = \sqrt{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(1000\text{ kg})/(9.8\text{ m/s}^2)} = 8.3 \times 10^{-5}$ m.

8. (A) is correct. The force due to gravity on an object at the Earth's surface is $F = m_o g = Gm_o m_e/r^2$, so $g = Gm_e/r^2$. Thus, above the Earth's surface, the value of $g = (6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\text{ kg})/(6.6 \times 10^6\text{ m})^2 = 9.16\text{ m/s}^2$.

9. (C) is the correct answer. Write an equation using Newton's law of universal gravitation with the force required to keep a mass in a uniform circular orbit, $Gm_s m_e/r^2 = m_s v^2/r$. Filling in values,
$(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\text{ kg})(1500\text{ kg})/(4.8 \times 10^7\text{ m})^2$
$= (1500\text{ kg})v^2/(4.8 \times 10^7\text{ m})$,
$v = \sqrt{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(5.98 \times 10^{24}\text{ kg})/(4.8 \times 10^7\text{ m})} = 2{,}900\text{ m/s}$.
Since $v = 2\pi r/T$, $T = 2\pi r/v = 2\pi(4.8 \times 10^7\text{ m})/2{,}900\text{ m/s} = 104{,}000\text{ s} =$ 29 hours.

10. (B) is correct. For planet 1, $r_1 = R$, and the frequency, f_1 is sought. Its period, consequently, is $T_1 = 1/f_1$. For planet 2, $r_2 = R/2$, and the frequency is 2.10×10^{-5} rev/s. Its period, consequently, is $T_2 = 1/(2.1 \times 10^{-5}\text{ rev/s}) =$ 47,600 seconds. We can apply Kepler's third law, $(T_1/T_2)^2 = (r_1/r_2)^3$ or $(T_1/47{,}600\text{ s})^2 = [R/(R/2)]^3$. Thus $T_1^2 = (2^3)(47{,}600\text{ s})^2 = 135{,}000$. Frequency $f = 1/T_1 = 7.4 \times 10^{-6}$ rev/s.

Free-Response Questions

1. (a) As the car is briefly in circular motion, its radial force would be given by $\Sigma F_R = mv^2/r = (750\text{ kg})(V^2)/35$ m. The force of static friction between the icy road and the tires, $\mu(750\text{ kg})(9.8\text{ m/s}^2)$, must be sufficient to keep the car on its circular path.

(b) As stated, the force of friction must provide the radial centripetal force, so $(750\text{ kg})(V^2)/35\text{ m} \leq \mu(750\text{ kg})(9.8\text{ m/s}^2)$. Therefore,

$$\frac{(750\text{ kg})(V^2)}{35\text{ m }(750\text{ kg})(9.8\text{ m/s}^2)} < \mu,$$

so

$$\mu > \frac{V^2}{35\text{ m }(9.8\text{ m/s}^2)} = \frac{V^2}{343\text{ m}^2/\text{s}^2}.$$

(c) If the coefficient of friction is 0.45, the force of friction is $(0.45)(750\text{ kg})(9.8\text{ m/s}^2) = 3308\text{ N}$, and $(750\text{ kg})(V^2)/35\text{ m} < 3308\text{ N}$, so V must be kept at or below 12.4 m/s.

This response correctly identifies the forces acting on the car as it travels in uniform circular motion, and it arranges the appropriate inequalities for variables. The response to part a sums up the thrust of the question. The response to part b uses a pure variable term to solve for another (V and μ), while the response to part c solves for an explicit value for the inequality.

2. For Kepler's third law, all objects in elliptical orbit around a body at one focus have a constant ratio of the cube of the orbital radius to the square of the period. This ratio could be provided by the given information about the satellite. Therefore, if either the Moon's orbital radius or its period were known, the other value could be determined—so only one of those two pieces of information is necessary. From there, the Newton's law of universal gravitation, $F = Gm_e m_m/r^2$, could set equal to the general force equation $F = m_m v^2/r$, where $v = 2\pi r/T$. Thus $F = Gm_e m_m/r^2 = m_m (2\pi r/T)^2/r$. This could be solved for the mass of the Earth without the mass of the Moon. In summation, only the Moon's orbital radius or period is necessary. It should be noted that the satellite's given information already would be sufficient to have estimated the Earth's mass.

   This answer details and utilizes Kepler's law, along with Newton's law of universal gravitation. The calculations demonstrate that only one variable is necessary, because Kepler's law can be used to find the other necessary variable for Newton's law. Similarly, the independence of the results from the mass of the Moon shows optimal understanding of the interrelation between both force equations.